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ISOMORPHISM OF CATEGORIES

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The following theorem is inspired by, and implies, the Cantor–Bernstein theorem. Both show that isomorphism, of sets or categories, is equivalent to an apparently weaker concept. The terminology and notations are as in [1].

Theorem. *Let $S:A\to C$ be an equivalence of categories with backwards functor $T:C\to A$ and natural isomorphisms $\eta:I\to ST$ and $\varepsilon:TS\to I$. If the functors S and T are injective for objects, then the categories A and C are isomorphic.*

Proof. Let $J_0=\mathrm{Ob}\,C\setminus S(\mathrm{Ob}\,C)$. For $n\geq 0$ let $J_n=(ST)^nJ_0$ and $E_n=TJ_n=T(ST)^nJ_0$. Let $J=\bigcup_{n=0}^{\infty}J_n$ and $E=\bigcup_{n=0}^{\infty}E_n$. Since $S:\mathrm{Ob}\,A\to\mathrm{Ob}\,C$ and $T:\mathrm{Ob}\,C\to\mathrm{Ob}\,A$ are injective, it follows from the definition of J_0 that the classes J_n are mutually disjoint, the classes E_n are mutually disjoint, T maps J bijectively to E, and S maps E bijectively to $J\setminus J_0$. Let $D=\mathrm{Ob}\,A\setminus E$ and $K=\mathrm{Ob}\,C\setminus J$. Then S maps D bijectively to K and T maps K into D.

For $a\in D$, let $Pa=Sa$ and let $f_a=1_{Sa}:Pa\to Sa$. For $a\in E$, let $Pa=T^{-1}a\in J$ and let $f_a=\eta_{Pa}:Pa\to STPa=Sa$. In either case $f_a:Pa\to Sa$ is an isomorphism.

For $c\in J$, let $Qc=Tc$ and let $g_c=1_{Tc}:Qc\to Tc$. For $c\in K$, let $Qc=S^{-1}c$ and let g_c be the inverse of $\varepsilon_{Qc}:TSS^{-1}c\to S^{-1}c$, so $g_c=\varepsilon_{Qc}^{-1}:Qc\to Tc$. In either case, $g_c:Qc\to Tc$ is an isomorphism.

For $a\in D$, $Tf_a=1_{TSa}$ and $g_{Pa}=g_{Sa}=\varepsilon_a^{-1}$. For $a\in E$, by the triangular identity $\varepsilon T\circ T\eta=1$ we have $\varepsilon_a\circ T\eta_{Pa}=1_a$ and hence $Tf_a=T\eta_{Pa}=\varepsilon_a^{-1}$. Also $g_{Pa}=1_{TPa}=1_a$. In either case $Tf_a\circ g_{Pa}=\varepsilon_a^{-1}$.

For $c\in J$, $Sg_c=1_{STc}$ and $f_{Qc}=f_{Tc}=\eta_c$. For $c\in K$, by the triangular identity $S\varepsilon\circ\eta S=1$ we have $S\varepsilon_{Qc}\circ\eta_c=1_c$ and hence $Sg_c=S\varepsilon_{Qc}^{-1}=\eta_c$. Also $f_{Qc}=1_c$. In either case $Sg_c\circ f_{Qc}=\eta_c$.

For each morphism $u:a\to b$ in A, let $Pu=f_b^{-1}\circ Su\circ f_a:Pa\to Pb$. For $v:c\to d$ in C, let $Qv=g_d^{-1}\circ Tv\circ g_c:Qc\to Qd$.

We are to show that P and Q are functors with $PQ=1$ and $QP=1$.

Let $a\in\mathrm{Ob}\,A$. Then

$$P(1_a)=f_a^{-1}\circ S(1_a)\circ S(1_a)\circ f_a=f_a^{-1}\circ 1_{Sa}\circ f_a=1_{Pa}.$$

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If $a \xrightarrow{u} b \xrightarrow{v} c$ in A,

$$P(v\circ u)=f_c^{-1}\circ S(v\circ u)\circ f_a=f_c^{-1}\circ S(v)\circ f_b\circ f_b^{-1}\circ S(u)\circ f_a=P(v)\circ P(u).$$

Thus $P: A\to C$ is a functor. Similarly $Q: C\to A$ is a functor.

Let $a\in \mathrm{Ob}\, A$. If $a\in D$, $QPa=QSa=S^{-1}Sa=a$, while if $a\in E$, $QPa=QT^{-1}a=TT^{-1}a=a$. Similarly if $c\in \mathrm{Ob}\, C$, $PQc=c$. If $u: a\to b$ in A,

$$\begin{aligned} QPu &= g_{Pb}^{-1}\circ TPu\circ g_{Pa}=g_{Pb}^{-1}\circ Tf_b^{-1}\circ TSu\circ Tf_a\circ g_{Pa}\\ &=\varepsilon_b\circ TSu\circ \varepsilon_a^{-1}=u. \end{aligned}$$

If $v: c\to d$ in C,

$$\begin{aligned} PQv &= f_{Qd}^{-1}\circ SQv\circ f_{Qc}=f_{Qd}^{-1}\circ Sg_d^{-1}\circ STv\circ Sg_c\circ f_{Qc}\\ &=\eta_d^{-1}\circ STv\circ \eta_c=v. \end{aligned}$$

Thus $Q\circ P: A\to A$ and $P\circ Q: C\to C$ are identity functors. Therefore $P: A\to C$ is an isomorphism of categories.

Note that since S and T are faithful [1], if they are injective for objects they must also be injective for morphisms.

In Pontryagin duality [2], [3], an equivalence is given by a contravariant functor P from the category **LCAb** of locally compact abelian groups to itself, where $P(G)$ is the character group of G. The backwards functor is P itself. Each element of $P(G)$ is a continuous homomorphism of G to the circle group R/Z, so each element of $P(G)$ determines G. Thus P is injective for objects.

Corollary 1. *The category* **LCAb** *of locally compact abelian groups is isomorphic to its opposite* **LCAb**$^{\mathrm{op}}$.

The character groups of compact abelian groups are discrete, and the character groups of discrete abelian groups are compact.

Corollary 2. *The category* **CAb** *of compact abelian groups is isomorphic to the opposite* **Ab**$^{\mathrm{op}}$ *of the category of abelian groups.*

References

[1] S. MacLane, Categories for the Working Mathematician (Springer, Berlin–New York, 1971).
[2] L.S. Pontryagin, Topological Groups (Princeton University Press, 1939).
[3] D.W. Roeder, Category theory applied to Pontryagin duality, Pacific J. Math. 52 (1974) 519–527.